# Numerické metódy
## riešenia elektromechanických úloh
## (časť 2)

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# Časť 2: Metóda konečných prvkov

Základné rovnice
Príprava systému
Spájanie elementov
Okrajové podmienky
Príklady
Nadstavbové aplikácie

Hodnotenie prednášky + študijné materiály:
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$$\nabla \cdot \kappa \nabla U + Q = 0 \rightarrow \kappa \nabla^2 U + Q = 0 \rightarrow \nabla^2 U = 0$$

## Okrajové podmienky:

$$U = U_0$$ Dirichlet

$$\frac{\partial U}{\partial n} = \frac{\partial U_0}{\partial n}$$ Neumann

$$aU_0 + b\frac{\partial U_0}{\partial n} = c$$

# Numerické metódy

## MKP – Základné rovnice

$$U = \sum_i N_i U_i$$

$$R = \left( \nabla . \kappa \nabla \sum_i N_i U_i + Q \right) = 0$$

$$w_j R = 0$$

$$\int_\Omega w_j R d\Omega = 0$$

Kolokačné metódy,
Galerkinova globálna metóda,
MKP,
a iné...

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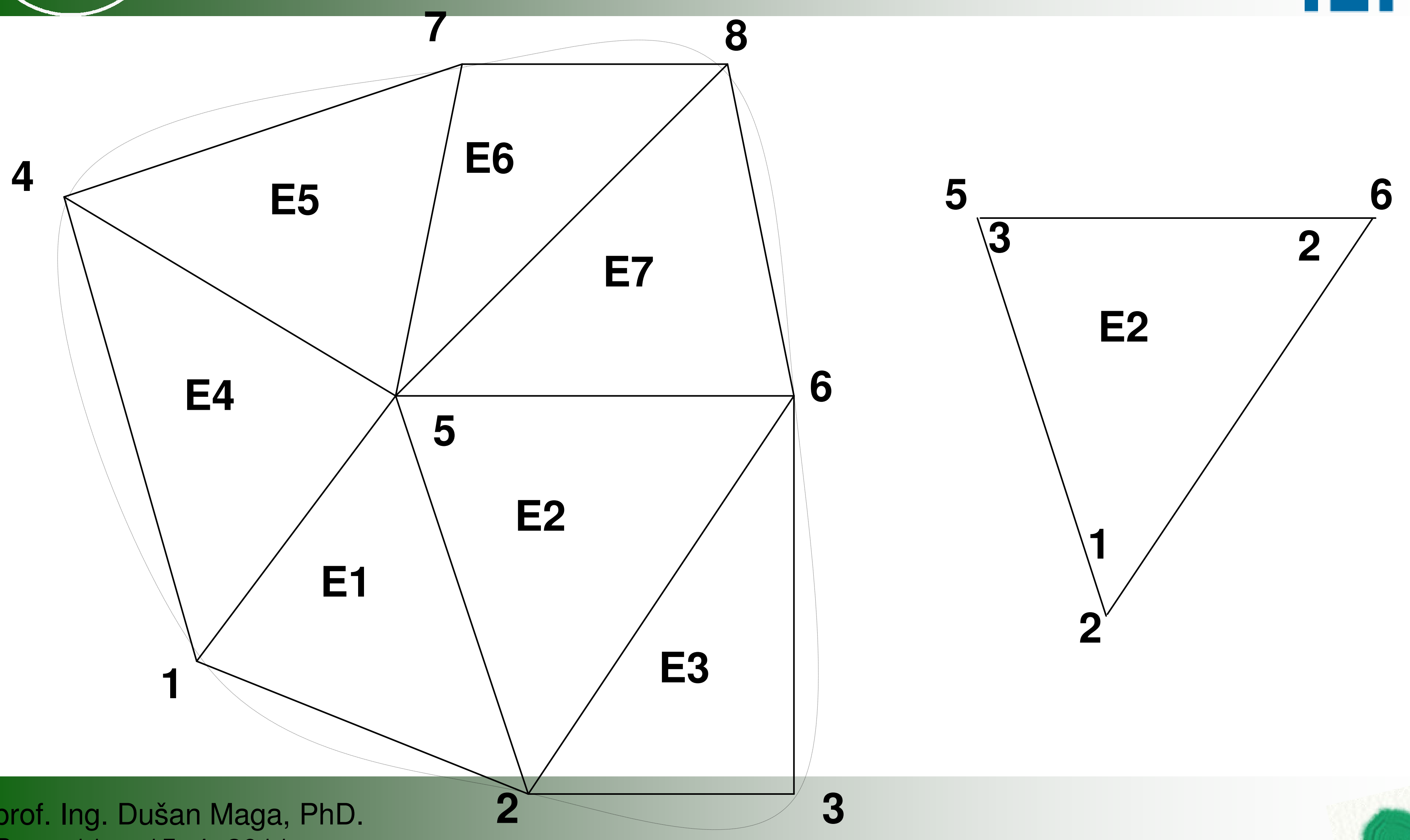
$$\int_{\Omega} w_j \left[ \nabla \cdot \kappa \nabla U + Q \right] d\Omega = 0$$

$$\int_{\Omega} d\Omega = \sum_{e=1}^{E} \int_{elem} d\Omega$$

$$\sum_{e=1}^{E} \int_{elem} w_j \left[ \nabla \cdot \kappa \nabla U + Q \right] d\Omega = 0$$

# Numerické metódy

## MKP – Príprava systému

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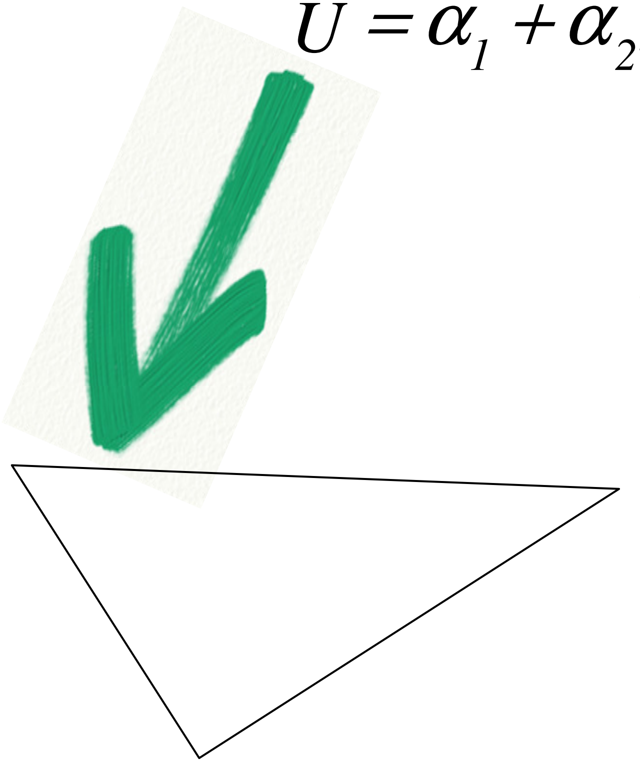
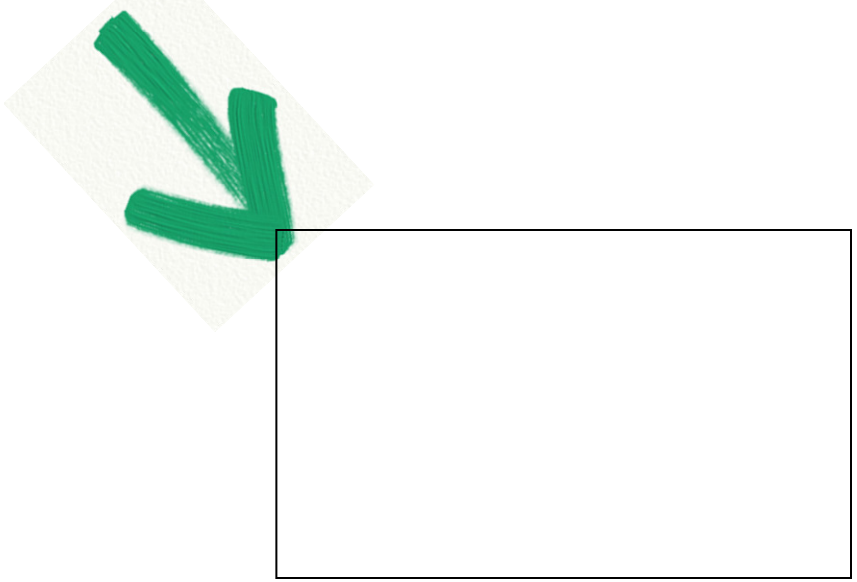
$$U = \alpha_1 + \alpha_2 x + \alpha_3 y$$



$$U = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 xy$$

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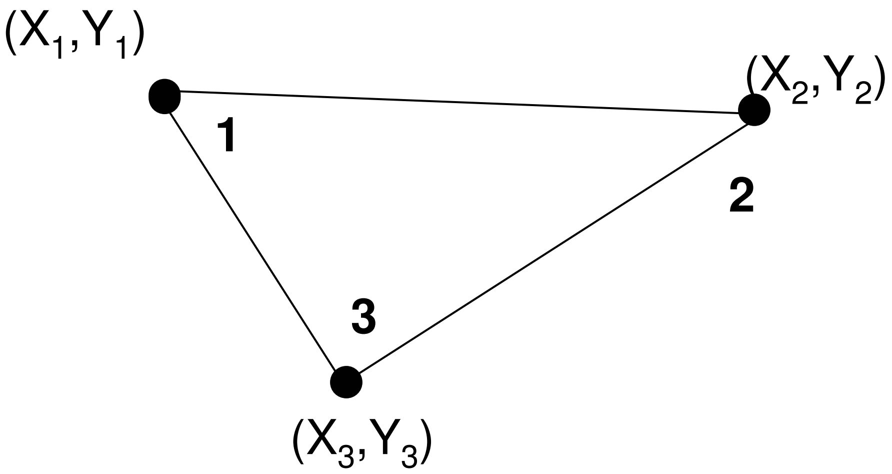
**Znalosti a dovednosti v mechatronice - transfer inovací do praxe, CZ.1.07/2.3.00/09.0162**

$$U_1 = \alpha_1 + \alpha_2 x_1 + \alpha_3 y_1$$

$$U_2 = \alpha_1 + \alpha_2 x_2 + \alpha_3 y_2$$

$$U_3 = \alpha_1 + \alpha_2 x_3 + \alpha_3 y_3$$





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# Numerické metódy

## MKP – Príprava systému

$$U_1 = \alpha_1 + \alpha_2 x_1 + \alpha_3 y_1$$

$$U_2 = \alpha_1 + \alpha_2 x_2 + \alpha_3 y_2$$

$$U_3 = \alpha_1 + \alpha_2 x_3 + \alpha_3 y_3$$

$$\begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}$$

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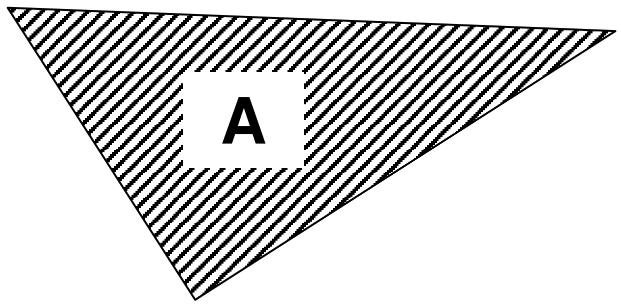
Riešenie:

$$\alpha_1 = \frac{1}{2A}\begin{vmatrix} U_1 & x_1 & y_1 \\ U_2 & x_2 & y_2 \\ U_3 & x_3 & y_3 \end{vmatrix}$$

$$\alpha_2 = \frac{1}{2A}\begin{vmatrix} 1 & U_1 & y_1 \\ 1 & U_2 & y_2 \\ 1 & U_3 & y_3 \end{vmatrix}$$

$$\alpha_3 = \frac{1}{2A}\begin{vmatrix} 1 & x_1 & U_1 \\ 1 & x_2 & U_2 \\ 1 & x_3 & U_3 \end{vmatrix}$$

$$2A = \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}$$





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Zjednodušujúce formálne značenie:

| | | |
|---|---|---|
| $a_1 = x_2y_3 - x_3y_2$ | $a_2 = x_3y_1 - x_1y_3$ | $a_3 = x_1y_2 - x_2y_1$ |
| $b_1 = y_2 - y_3$ | $b_2 = y_3 - y_1$ | $b_3 = y_1 - y_2$ |
| $c_1 = x_3 - x_2$ | $c_2 = x_1 - x_3$ | $c_3 = x_2 - x_1$ |

$$\alpha_1=(a_1U_1+a_2U_2+a_3U_3)/2A$$

$$\alpha_2=(b_1U_1+b_2U_2+b_3U_3)/2A$$

$$\alpha_3=(c_1U_1+c_2U_2+c_3U_3)/2A$$

$$U=\alpha_1+\alpha_2x+\alpha_3y$$

$$U = \alpha_1 + \alpha_2 x + \alpha_3 y$$

$$U=1/(2A)[(a_1+b_1x+c_1y)U_1+(a_2+b_2x+c_2y)U_2+(a_3+b_3x+c_3y)U_3$$

$$U = \sum_i \left[ \frac{1}{2A} (a_i + b_i x + c_i y) U_i \right]$$

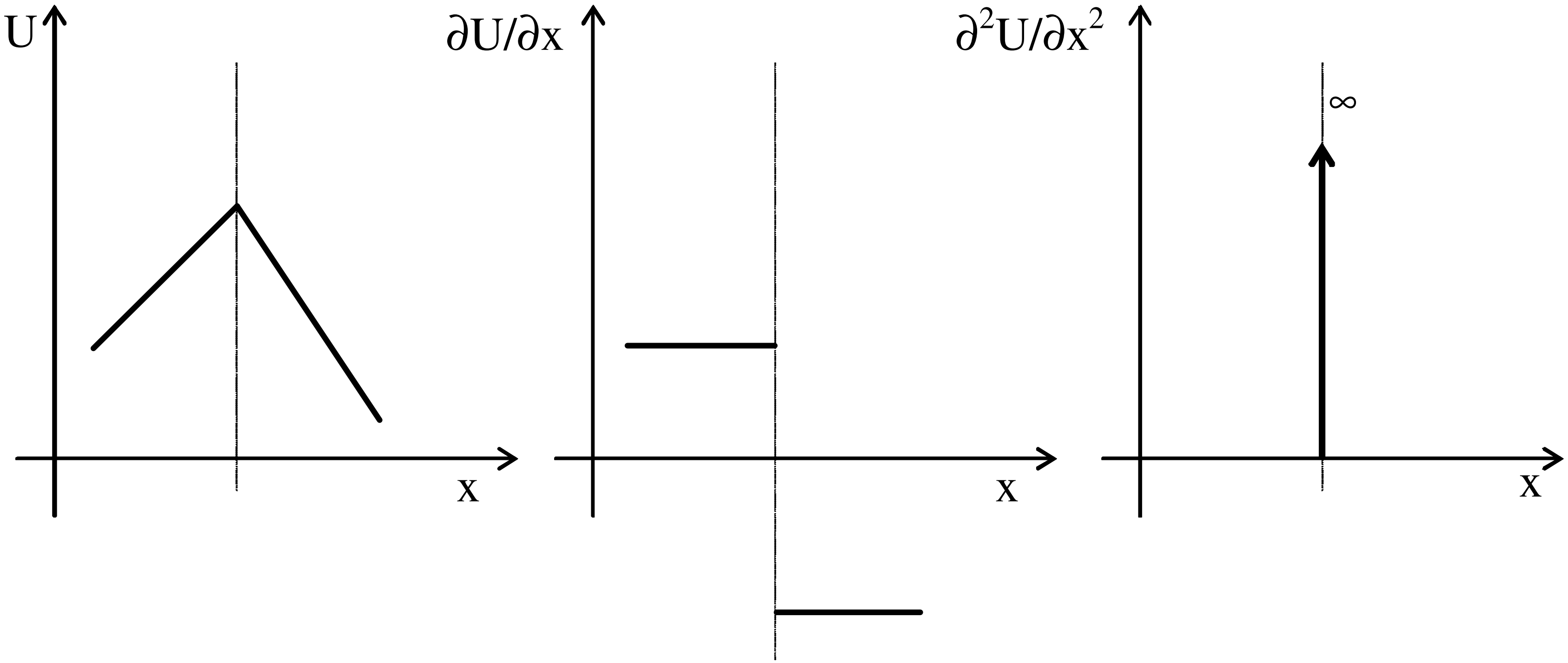
$$U = \sum_i \left[ \frac{1}{2A} (a_i + b_i x + c_i y) U_i \right]$$

$$N_i = \frac{a_i + b_i x + c_i y}{2A}$$

Platí aj:

$$N_i(x_k, y_k) = \delta_{ik}$$

Problémy:

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Odstránenie druhej derivácie:

$$\int_\Omega \nabla w \cdot \nabla U d\Omega + \int_\Omega w \cdot \nabla^2 U d\Omega = \int_\Gamma w \nabla U . d\Gamma$$

(Gauss-Ostrogradského teoréma)

$$\int_\Omega w_j [\nabla \cdot \kappa \nabla U + Q] d\Omega =$$

$$= \int_\Omega \nabla w_j \cdot \kappa \nabla U d\Omega + \int_\Omega w_j Q d\Omega - \int_\Gamma w_j \kappa \frac{\partial U}{\partial n} d\Gamma = 0$$

Preformulovanie podľa metodiky MKP + váhové funkcie podľa Galerkina ($w_j = N_j$):

$$R_i = \sum_j \left[ \left( \int_{elem} \nabla N_i \kappa \nabla N_j d\Omega \right) U_j \right] + \int_{elem} N_i Q d\Omega$$

$$R_i = k_{ij} U_j + f_i$$


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$$R_i = k_{ij} U_j + f_i$$

$$k_{ij} = \int_{elem} \kappa \left( \frac{\partial N_i}{\partial x} \frac{\partial N_j}{\partial x} + \frac{\partial N_i}{\partial y} \frac{\partial N_j}{\partial y} \right) d\Omega$$

$$f_i = \int_{elem} N_i Q d\Omega$$


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$$R_i = k_{ij} U_j + f_i$$

$$k_{ij} = \int_{elem} \frac{\kappa\left(b_i b_j + c_i c_j\right)}{4A^2} d\Omega = \frac{\kappa\left(b_i b_j + c_i c_j\right)}{4A}$$

$$f_i = \int_{elem} \frac{\left(a_i + b_i x + c_i y\right)}{2A} Q d\Omega = \frac{AQ}{3}$$

$$\begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix} = \frac{\kappa}{4A} \begin{bmatrix} b_1^2 + c_1^2 & b_1 b_2 + c_1 c_2 & b_1 b_3 + c_1 c_3 \\ b_1 b_2 + c_1 c_2 & b_2^2 + c_2^2 & b_2 b_3 + c_2 c_3 \\ b_1 b_3 + c_1 c_3 & b_2 b_3 + c_2 c_3 & b_3^2 + c_3^2 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} + \frac{AQ}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

*b1 = y2-y3*
*c1 = x3-x2*

*b2 = y3-y1*
*c2 = x1-x3*

*b3 = y1-y2*
*c3 = x2-x1*


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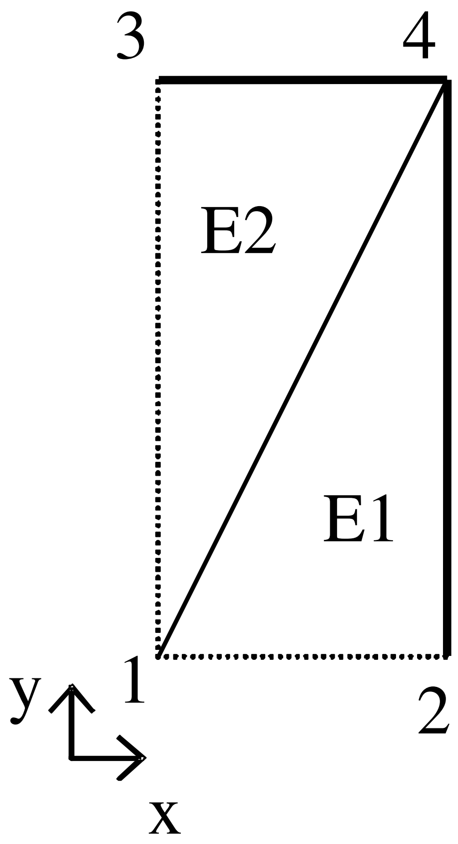
# Numerické metódy

## MKP – Spájanie elementov

$$k_{ij}^{(n)} = \frac{\kappa}{4A}\left(b_i^{(n)} b_j^{(n)} + c_i^{(n)} c_j^{(n)}\right)$$

$$f_i^{(n)} = \frac{AQ}{3}$$

$$\begin{bmatrix} k_{11}^{(1)} + k_{11}^{(2)} & k_{12}^{(1)} & k_{13}^{(2)} & k_{14}^{(1)} + k_{14}^{(2)} \\ k_{12}^{(1)} & k_{22}^{(1)} & 0 & k_{24}^{(1)} \\ k_{13}^{(2)} & 0 & k_{33}^{(2)} & k_{43}^{(2)} \\ k_{14}^{(1)} + k_{14}^{(2)} & k_{24}^{(1)} & k_{43}^{(2)} & k_{44}^{(1)} + k_{44}^{(2)} \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{bmatrix} = -\begin{bmatrix} f_1^{(1)} + f_1^{(2)} \\ f_2^{(1)} \\ f_3^{(2)} \\ f_4^{(1)} + f_4^{(2)} \end{bmatrix}$$

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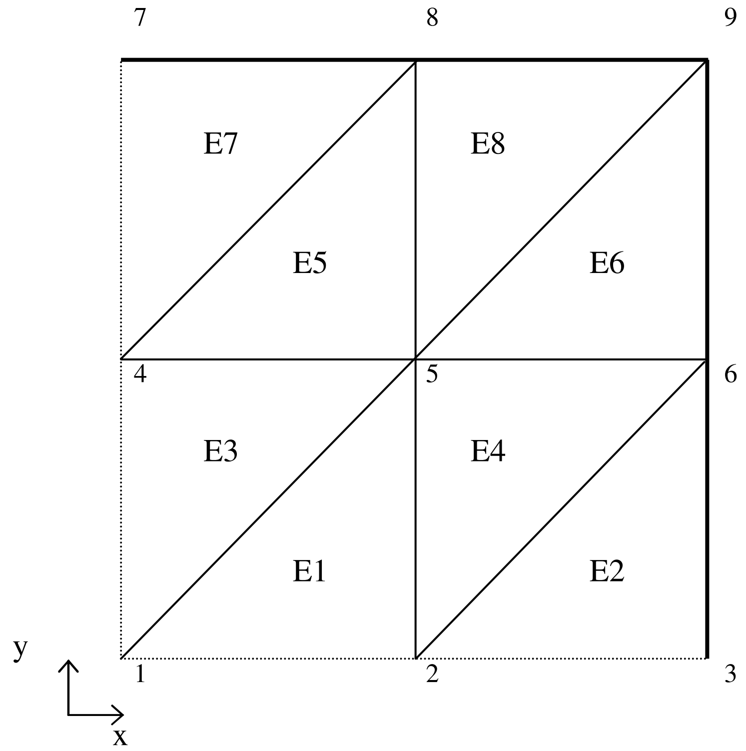
$$\begin{bmatrix} k_{11} & k_{12} & 0 \\ k_{21} & k_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} = \begin{bmatrix} f_1 - k_{13} u_3 \\ f_2 - k_{23} u_3 \\ u_3 \end{bmatrix}$$

| | *element č.:* | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | $E_1$ | $E_2$ | $E_3$ | $E_4$ | $E_5$ | $E_6$ | $E_7$ | $E_8$ |
| $b_1$ | -1 | -1 | 0 | 0 | -1 | -1 | 0 | 0 |
| $b_2$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $b_3$ | 0 | 0 | -1 | -1 | 0 | 0 | -1 | -1 |
| $c_1$ | 0 | 0 | -1 | -1 | 0 | 0 | -1 | -1 |
| $c_2$ | -1 | -1 | 0 | 0 | -1 | -1 | 0 | 0 |
| $c_3$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |





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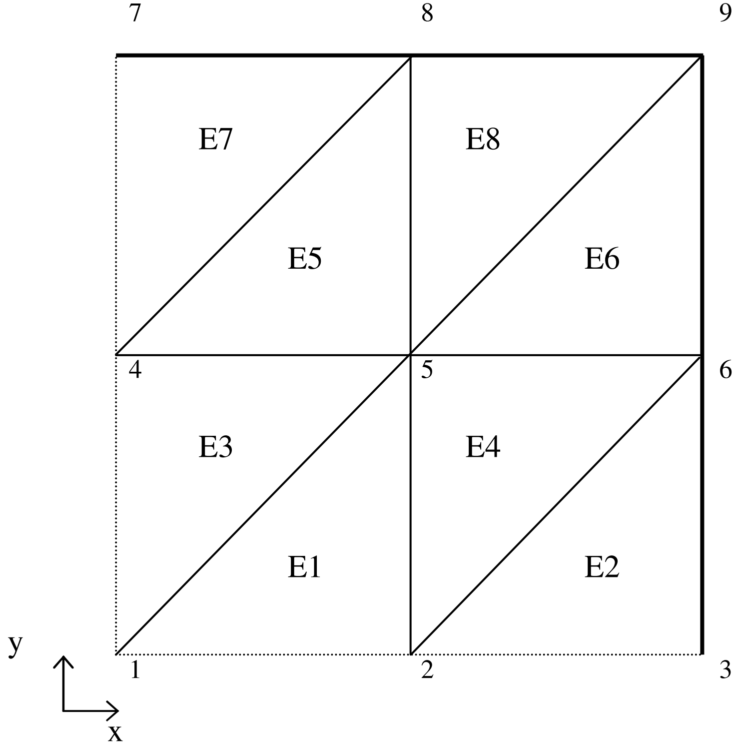
E1:

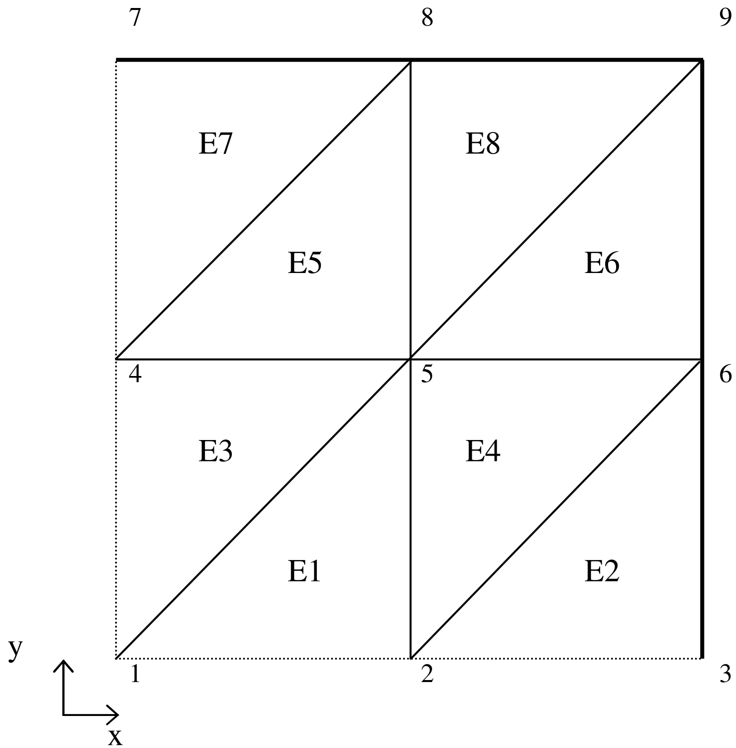
$$\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_5 \end{bmatrix}$$

E3:

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} U_1 \\ U_5 \\ U_4 \end{bmatrix}$$



$$
\begin{bmatrix}
2 & -1 & & -1 & 0 & & & & \\
-1 & 4 & -1 & & -2 & 0 & & & \\
 & -1 & 2 & & & -1 & & & \\
-1 & & & 4 & -2 & & -1 & 0 & \\
0 & -2 & & -2 & 8 & -2 & & -2 & 0 \\
 & 0 & -1 & & -2 & 4 & & & -1 \\
 & & & -1 & & & 2 & -1 & \\
 & & & 0 & -2 & & -1 & 4 & -1 \\
 & & & & 0 & -1 & & -1 & 2
\end{bmatrix}
\begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \\ U_6 \\ U_7 \\ U_8 \\ U_9 \end{bmatrix}
=
\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$





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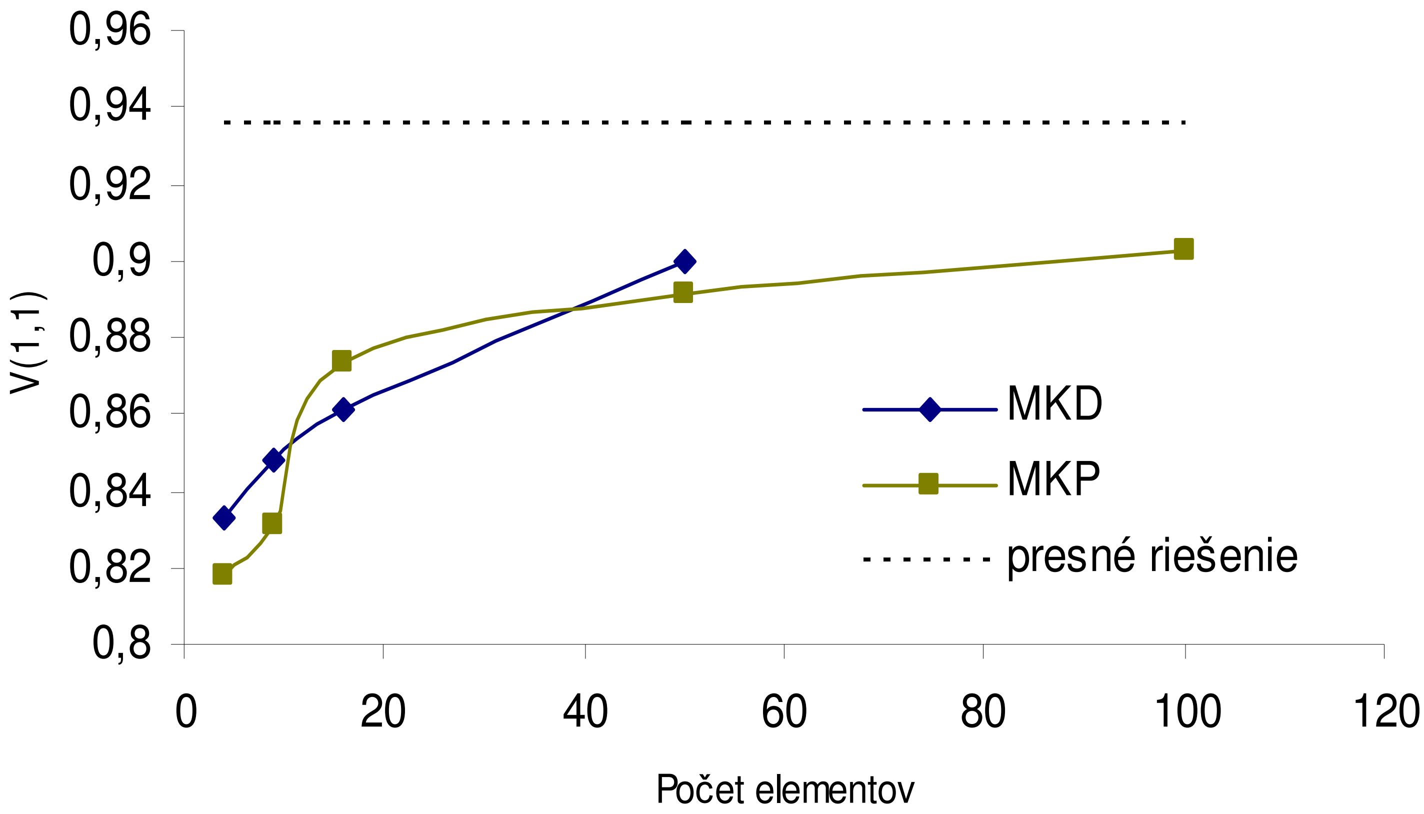
# Numerické metódy

## MKP – Príklady

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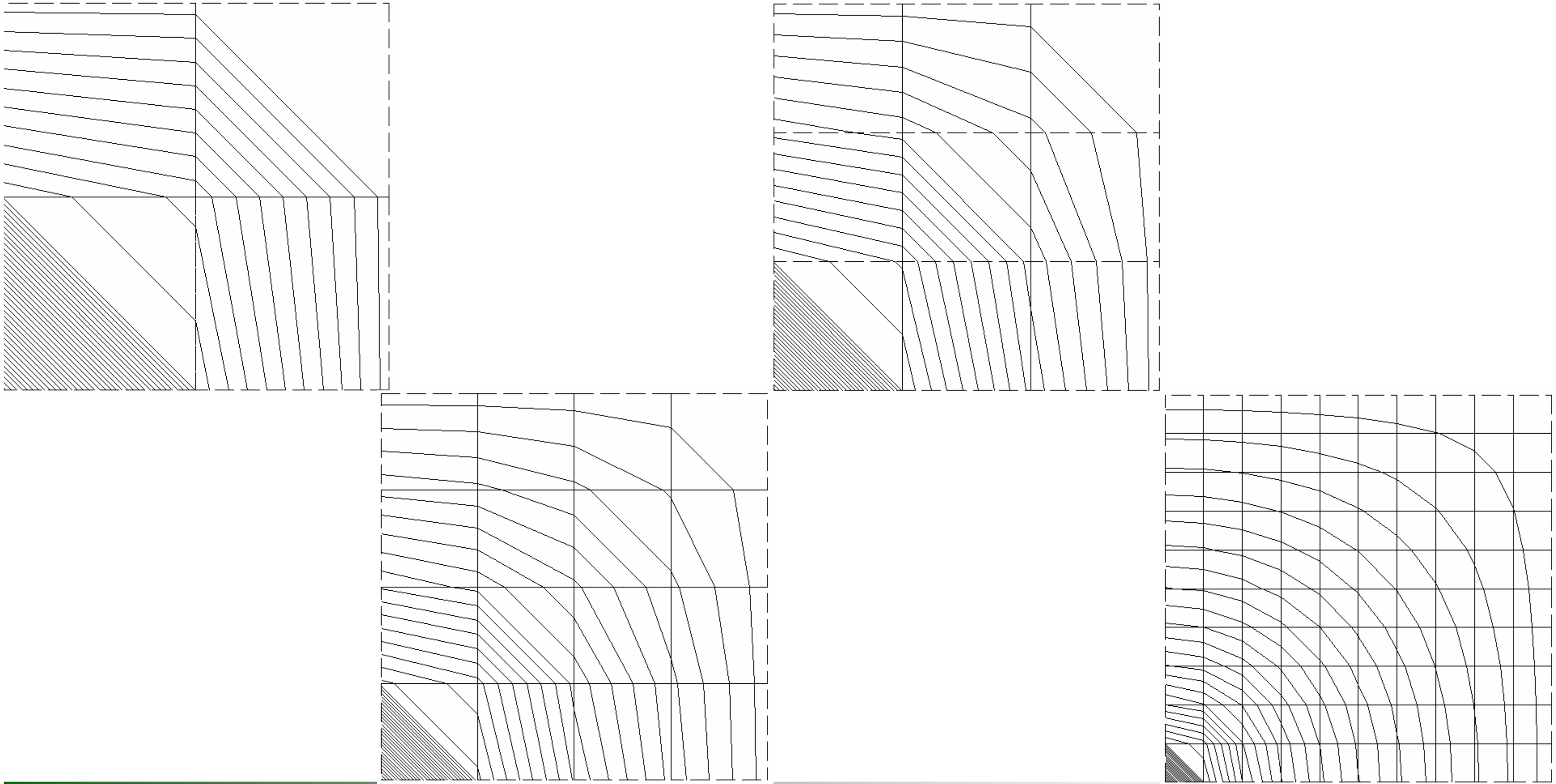
## MKP – Príklady



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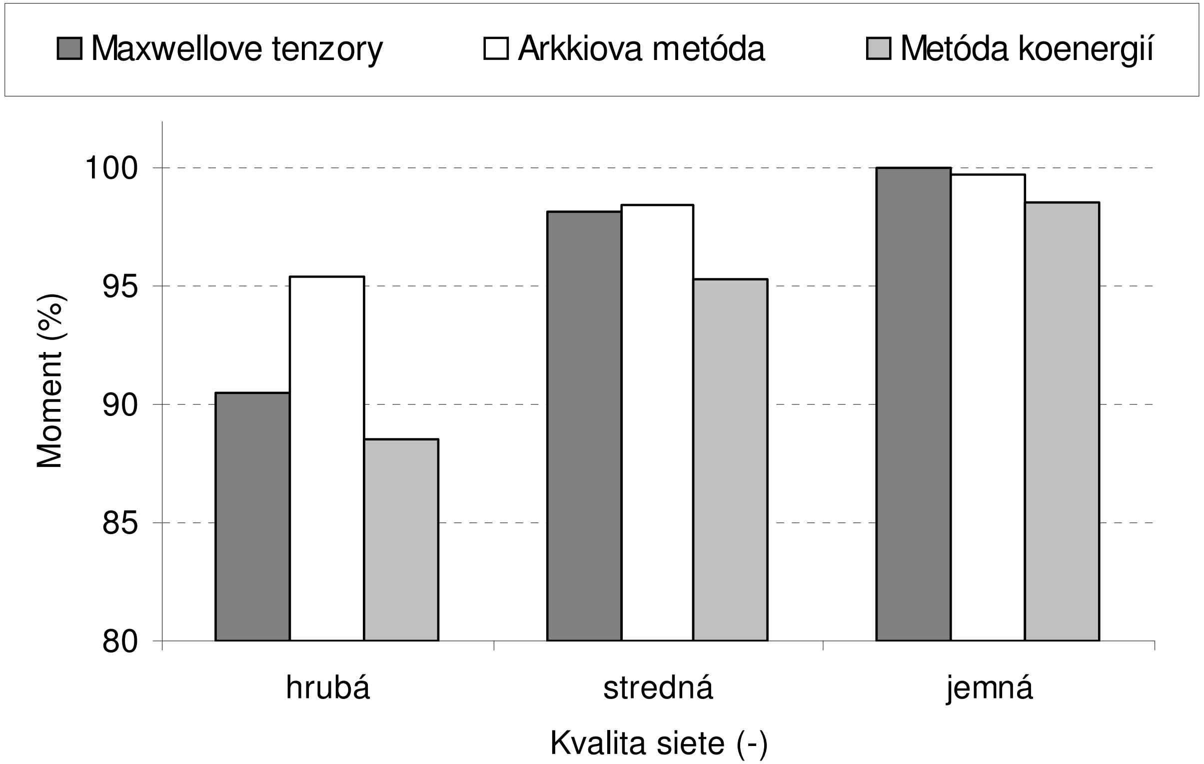
**Znalosti a dovednosti v mechatronice - transfer inovací do praxe, CZ.1.07/2.3.00/09.0162**

## Metódy výpočtu momentov:

- metóda Maxwellových tenzorov, $$M = \frac{1}{\mu_0} \oint_{\Gamma} r B_t B_n \, d\Gamma$$
- Arkkiova metóda (tiež aj metóda spriemerovaných Maxwellových tenzorov), $$M = \frac{1}{\mu_0 \delta} \iint_S r B_t B_n dS$$
- metóda koenergií, $$W' = \iint_S \left( \int_0^H b dh \right) dS$$ $$W'' = \iint_S \left( \int_0^H b dh \right) dS$$ $$M = \frac{W'' - W'}{\Delta\theta} = \frac{\Delta W}{\Delta\theta}$$
- metóda virtuálnych prác.

## Metódy výpočtu momentov:



## Časovo závislé deje:

$$\begin{bmatrix} b_1^2+c_1^2-2j\frac{A^2k\omega}{3\kappa} & b_1b_2+c_1c_2-j\frac{A^2k\omega}{3\kappa} & b_1b_3+c_1c_3-j\frac{A^2k\omega}{3\kappa} \\ b_2b_1+c_2c_1-j\frac{A^2k\omega}{3\kappa} & b_2^2+c_2^2-2j\frac{A^2k\omega}{3\kappa} & b_2b_3+c_2c_3-j\frac{A^2k\omega}{3\kappa} \\ b_3b_1+c_3c_1-j\frac{A^2k\omega}{3\kappa} & b_3b_2+c_3c_2-j\frac{A^2k\omega}{3\kappa} & b_3^2+c_3^2-2j\frac{A^2k\omega}{3\kappa} \end{bmatrix} \begin{bmatrix} U_1^C \\ U_2^C \\ U_3^C \end{bmatrix} = -\frac{4}{3}\frac{A^2}{\kappa}Q^C \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

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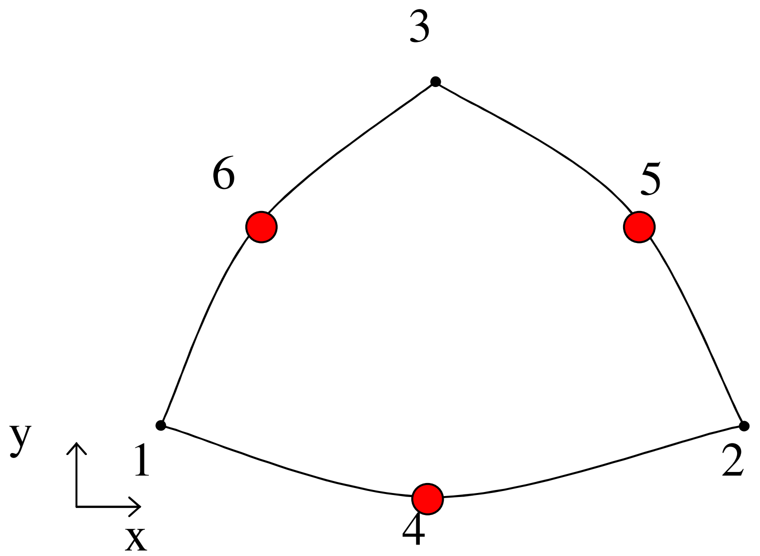
## Elementy vyšších rádov:

$$U = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 xy + \alpha_5 x^2 + \alpha_1 y^2$$

$$\begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \\ U_6 \end{bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 & x_1 y_1 & x_1^2 & y_1^2 \\ 1 & x_2 & y_2 & x_2 y_2 & x_2^2 & y_2^2 \\ 1 & x_3 & y_3 & x_3 y_3 & x_3^2 & y_3^2 \\ 1 & x_4 & y_4 & x_4 y_4 & x_4^2 & y_4^2 \\ 1 & x_5 & y_5 & x_5 y_5 & x_5^2 & y_5^2 \\ 1 & x_6 & y_6 & x_6 y_6 & x_6^2 & y_6^2 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \end{bmatrix}$$

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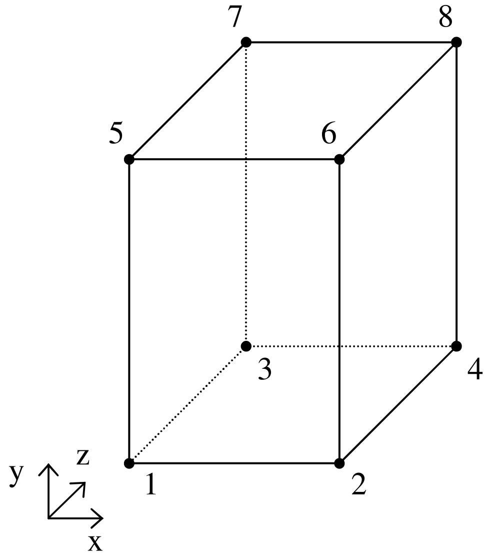
3D:

$$U = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 z + \alpha_5 xy + \alpha_6 yz + \alpha_7 zx + \alpha_8 xyz$$

$$\begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \\ U_6 \\ U_7 \\ U_8 \end{bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 & z_1 & x_1y_1 & y_1z_1 & z_1x_1 & x_1y_1z_1 \\ 1 & x_2 & y_2 & z_2 & x_2y_2 & y_2z_2 & z_2x_2 & x_2y_2z_2 \\ 1 & x_3 & y_3 & z_3 & x_3y_3 & y_3z_3 & z_3x_3 & x_3y_3z_3 \\ 1 & x_4 & y_4 & z_4 & x_4y_4 & y_4z_4 & z_4x_4 & x_4y_4z_4 \\ 1 & x_5 & y_5 & z_5 & x_5y_5 & y_5z_5 & z_5x_5 & x_5y_5z_5 \\ 1 & x_6 & y_6 & z_6 & x_6y_6 & y_6z_6 & z_6x_6 & x_6y_6z_6 \\ 1 & x_7 & y_7 & z_7 & x_7y_7 & y_7z_7 & z_7x_7 & x_7y_7z_7 \\ 1 & x_8 & y_8 & z_8 & x_8y_8 & y_8z_8 & z_8x_8 & x_8y_8z_8 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \\ \alpha_7 \\ \alpha_8 \end{bmatrix}$$

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Nelinearity v 2D poliach:

$$\kappa^{(1)} = f\left(U^{(0)}\right)$$

$$U^{(m+1)} = K^{-1}\left(U^{m}\right)Q$$

$$U^{(m+1)} - U^{(m)} = err$$

$$|err| < \alpha$$